BEBR

FACULTY WORKING PAPER NO. 859

Growth and the Levered Firm

Jonathan A. K. Cave

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

BEBR

FACULTY WORKING PAPER NO. 859

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

April 1982

"Growth and the Levered Firm"

Jonathan A. K. Cave, Assistant Professor
Department of Economics

Abstract

The Shapley value for nonatomic games is used to determine the shares of equity and risky debt in the value of a firm facing uncertain growth opportunities. We include the "voting power" attached to equity holdings and focus on the bargaining between shareholders and debt-holders, where the two classes may overlap, obtaining an extension of the Modigliani-Miller theorem.

Growth and the Levered Firm

I. Introduction

In this paper we take a game theoretic approach to the analysis of investment behavior in levered firms facing opportunities for growth. Despite the negative conclusions of the Modigliani-Miller theorem and its variants as to the meaning of corporate financial policy and the effect of corporate debt policy on optimal investment in a static world, Myers (1977) argues that the presence of risky debt maturing after investment decisions are taken provides a disincentive to the full exploitation of growth opportunities. This happens because the debt is risky: if the investment is made, its fruits must be shared with the debt-holders, but if it is not made, no payment need be made to them. Recently Aivazian and Callen (1980) took issue with Myers' results and provided a game-theoretic model which takes explicit account of the "negotiation risk" involved between the debt- and equity-holders. The basic idea is that the disincentive effect of the debt creates a loss in total value to the levered firm. By agreeing on a division of the inframarginal gains created by taking the optimal investment activity as perceived by the all-equity firm, the debt- and equity-holders can avoid this loss. The essential question is how this bargaining is to take place.

In the A-C model, the fact that there is both objective and strategic uncertainty lead the authors to propose using the Shapley value to allocate the inframarginal gains. However, there are two features of their application that limit the applicability of their results. In the first place, they formulate the bargaining as taking place between two separate players; an "equity player" and a "debt player". In the second place,

the amount of investment is not limited by financing possibilities, nor is it influenced by the state of nature.

The model we shall develop in this paper avoids these limitations. By using the Shapley value for nonatomic games, we are able to deal with a more realistic process of negotiation between all the parties, who may have differing amounts of debt and equity in their portfolios, and thus a wide spectrum of interests. In fact, we shall describe a "coalition" of parties by two measures, one of which is the percentage of the firm's equity held by members of the coalition, while the other is the percentage of the firm's debt held by coalition members. Moreover, we shall further refine the game by taking into account the "voting power" of equity, in the sense that a coalition with a majority of the firm's equity can vote for any feasible investment plan, while a minority coalition cannot.

As far as the level of investment is concerned, we shall present several formulations. In the first, we shall assume that the amount of investment that the all-equity firm would wish to engage in is sensitive to the state of nature, but does not depend on financing arrangements; the money is raised from outside sources, and repaid to those sources. In this case, the effect of debt is relatively weak: what matters is whether a coalition can vote for investment or not, and the only effect of debt-holdings is to reduce the winning coalition's debt liability to the rest of the players. In the second, we shall take the position that the investment funds are to be drawn from the company's debt, so that a winning coalition which does not have much debt may be limited in the scale of investment it undertakes by a financial constraint which may

be tighter than that provided by the state dependence of the value-of-investment function.

In the second section of the paper, we present the Myers and A-C approaches. The more general model is developed in the following three sections; first, we introduce the nonatomic value and calculate the implicit value of debt and equity when there is a fixed optimal scale of investment. Then we weaken this to allow for state dependence of optimal scale, and finally we present the model with a financial constraint as well. In the last section, we develop some game-theoretic subtleties and provide some suggestions for future exploitation of this model. The author wishes to thank C-F Lee for introducing him to the issues, and Y. Taumann and L. Mirman for helpful discussions on the topic. Responsibility for all errors remains with the author.

II. The model and previous results

The scenario we are dealing with is best described as follows: at date 0, a firm with some risky debt in its portfolio faces an investment decision. The state of the world is revealed at date 1, and the firm decides whether or not to invest a fixed amount. At date 2, if the firm has invested I and if the true state of the world revealed at $t = 1$ was S, the firm reaps a value V(S), pays the amount I to the banks, and pays the amount P to its debt-holders. On the other hand, if the firm did not invest at date 1, then all the parties receive exactly 0. Our measure of the value of the firm will be the present value of the firm's policy as of date 0; before the state is revealed.

We shall assume that all states have equal prior probability, and so that the value-of-investment function V(S) is increasing in S.

2.1 The all-equity firm

We can easily describe the value of the all-equity firm. Letting $q(S)$ be the equilibrium price at $t=0$ of one dollar at $t=1$ contingent on the occurrence of state S, the value of the optimal policy is

$$V^* = \int q(S)\max[0,\ V(S)-I]dS$$

We shall further define the "breakeven" state for the all equity firm, S_a, by

$$V(S_a) = I$$

so that $V^* = \int_{S}^{1} q(S)[V(S) - I]dS$, since the state space will be taken to be the unit interval, $[0,1]$.

2.2 The levered form

In this case, the firm has an obligation to pay \$P in period 2 if it undertakes the investment project. Thus, the firm's net disbursements in the event of investment are I to the banks and P to the debt-holders, so we may define the "breakeven" state for the levered firm, S_b, by

$$V(S_b) = I + P$$

Clearly, S_a is no greater than S_b. The values at $t = 0$ of the firm's debt [subscript B] and equity [subscript E] are thus:

$$V_B = P\int_{S_b}^{1} q(S)dS$$

$$V_E = \int_{S_b}^{1} q(S)V(S)dS - (I + P)\int_{S_b}^{1} q(S)dS$$

The total value of the levered firm under this "noncooperative" optimization is therefore $V_B + V_E = \int_{S_b}^{1} q(S)[V(S)-I]dS$, and the welfare loss due to the presence of the risky debt in the portfolio of the levered firm is:

$$L = \int_{S_a}^{S_b} q(S)[V(S) - I]dS$$

The next step in the argument is to posit a process of negotiation whereby the equity holders, in effect, buy out the debt-holders and undertake the optimal amount of investment. The price they pay lies somewhere between 0 and the amount of loss L. The result of this "Coasian" argument is that the firm follows the "correct" investment policy, resulting in a present value of V*. However, the allocation of the gain L between the two sides depends on their relative bargaining strengths.

To resolve this ambiguity, and to focus attention on the process of negotiation, A-C adopt three axioms from Roth's (1979) probabilistic approach to the Shapley value. These actions describe the value U(i,v) of playing position i in the game v. v is a characteristic function game of transferrable utility, which means that v is a map assigning to each possible coalition C a real number v(C) called the "worth" of C, representing a profit divisible between the members of C in any way they choose. These axioms state that: i) a person should be indifferent between games where he contributes nothing to any coalition; ii) a person should be risk-neutral as regards probabilistic combinations of games; and iii) a person should exhibit "strategic risk neutrality" in the sense that he is indifferent between the option of playing in a symmetric pure

bargaining game with k participants and receiving 1/k of the total to be divided in that game. The unique function U(i,v) satisfying these axioms is the Shapley value, which pays each player his expected marginal contribution to a "random coalition" consisting of the players preceeding the given player in a random ordering of the players drawn with probability $\frac{1}{N!}$.

The characteristic function used by A-C is for a two player game, with players E and B and is as follows:

$$\left.\begin{aligned} v(E:S) &= v(B:S) = 0 \\ v(E,B:S) &= \max[0, V(S) - I] \end{aligned}\right\} \quad \text{for } S \le S_b$$

and

$$\left.\begin{aligned} v(E:S) &= V(S) - I - P \\ v(B:S) &= P \\ v(E,B:S) &= V(S) - I \end{aligned}\right\} \quad \text{otherwise}$$

From these, we can calculate the Shapley value for each state S, and take its expected present value as of t = 0, giving

$$U(E) = \frac{1}{2} \int_{S_a}^{S_b} q(S)[V(S) - I]dS + \int_{S_b}^{1} q(S)[V(S) - I - P]dS$$

$$U(B) = \frac{1}{2} \int_{S_a}^{S_b} q(S)[V(S) - I]dS + P \int_{S_b}^{1} q(S)dS$$

In other words, the Shapley value in this case merely splits the difference in values between the levered and unlevered firms equally between the two partners E and B, and leaves the other gains untouched.

III. The more general model, I: portfolio effects

We regard the participants as forming a continuum, identified with the unit interval [0,1]. The set of possible coalitions is identified with the Borel subsets of the player set. Each coalition C has a share $E(C) \in [0,1]$ of the firm's equity and a share $B(C) \in [0,1]$ of the firm's debt. For the purposes of this section we shall fix the optimal amount of investment at I, regardless of the state. We do not need to assume that the two measures, E and B are independent, all we need is that the range of (E,B) is convex and compact, which follows from the assumption that they are nonatomic by Lyapunov's theorem.

Suppose that a coalition C has formed. If $E(C) < \frac{1}{2}$, then C is powerless to vote for new investment, and thus has a worth of 0. If C is a majority coalition and votes to undertake the investment project, its total payoff is

$$r(C) = [V(S) - I - P]E(C) + PB(C)$$

We can define the "breakeven" state for the coalition C, S_C, by

$$V(S_C) = I + P(1 - \frac{B(C)}{E(C)})$$

this being the state at which C will just find it worthwhile to invest. The characteristic function giving the worth of C in state S, w(C:S), is given by

$$w(C:S) = \begin{cases} 0 \text{ if } E(C) < 1/2 \text{ or } S < S_C \\ r(C) \text{ otherwise.} \end{cases}$$

Now r is a linear function of the two measure E and B, for fixed S, and so it belongs to the space of nonatomic games called DIAG: games in this space have the convenient property (which motivates our use of the continuum model) that their values are completely determined by the behavior of the characteristic function in a neighborhood of the "diagonal", where E(C) = B(C). In fact, we can make use of the "diagonal formula:" the value of a coalition C is just the integral up the diagonal of the directional derivation of the characteristic function in the direction (E(C),B(C)).

Of course, this is not quite the end of the story, since we are not working with the characteristic function r but with its "E-truncation." However, the value of a diagonal game which has a finite number of jumps across manifolds transverse to the diagonal still exists and can be easily calculated. In this case, if we let r(E,B) denote the function r defined above, evaluated at any C s.t. E(C) = E and B(C) = B, the value of the truncated game is:

$$U(C:S) = r(\tfrac{1}{2},\tfrac{1}{2})E(C) + \int_{1/2}^{1} \partial_C r(t,t)\,dt$$

where $\partial_C r$ denotes the directional derivative of r in the direction (E(C),B(C)):

$$\partial_C r(t,t) = [V(S) - I - P]E(C) + PB(C).$$

In other words, the value is given by:

$$U(C:S) = [V(S) - I]E(C) - \tfrac{P}{2}E(C) + \tfrac{P}{2}B(C)$$

at least for states $S \geq S_C$. However, thanks to the diagonal formula, the complication posed by the breakeven state does not affect us. The breakeven state for a diagonal coalition is exactly S_a as defined in Section 2.1 above.

To facilitate interpretation of this result, we must examine its efficiency, its rationality, and compare it to the A-C result for the two-player game. First, the worth of the grand coalition is precisely

$$U(N:S) = V(S) - I \qquad \text{for all } S \geq S_a$$

which shows efficiency. By "rationality" we mean the idea that no coalition is given less than it can assure itself of by independent action. In essence, this means that we are looking for an outcome in the core. However, due to the truncation of this game, the characteristic function is not convex, so we are not guaranteed that the value will be in the core, or indeed that the core exists at all. However, in this game the linearity of the untruncated game comes to our rescue: the core of the truncated game includes the core of the untruncated game, together with all allocations giving less than r(C) to minority coalitions. However, it can readily be seen that the value does not provide a core allocation: coalitions having more debt than equity are disadvantaged by the value.

Before comparing the result for the truncated game with the A-C result, we shall also determine the value of the game r by itself (without regard to the political dimension added by considering only coalitions with a majority of the firm's equity), and express both of these values as t=0 expected present values. The value of r is equal to r, since r is linear in both measures E and B. To facilitate expressing these values, let us define:

$$G(S) = \int_S^1 q(S)[V(S) - I]dS$$

$$Q(S) = \int_S^1 q(S)dS$$

Moreover, we shall "linearize" the A-C result by dividing U(E) proportionately to E(C) and U(B) proportionately to B(C). Our three values are:

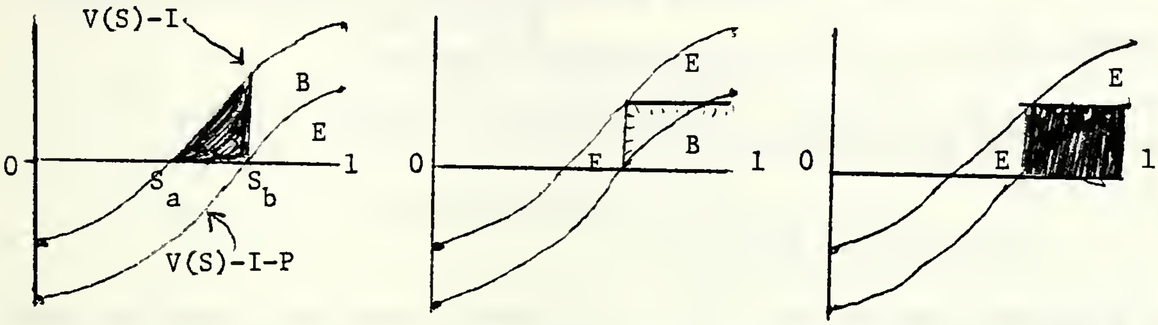
$$\text{A-C: } U(C) = \tfrac{1}{2}[G(S_a)+G(S_b)-PQ(S_b))]E(C) + \tfrac{1}{2}[G(S_a)-G(S_b)+PQ(S_b)]B(C)$$

$$\text{untruncated r: } r(C) = [G(S_a)-PQ(S_a)]E(C) + [PQ(S_a)]B(C)$$

$$\text{truncated r: } W(C) = [G(S_a) - \tfrac{1}{2}PQ(S_a)]E(C) + \tfrac{1}{2}[PQ(S_a)]B(C)$$

Several words of interpretation are in order at this point. First, both the untruncated game and the linearized A-C result have values in the core, while the truncated game does not. Further insight comes from comparing the shadow prices of debt and equity. Both U and W divide the scheduled debt payment $PQ(S_b)$ for states greater than S_b equally between debt and equity. However, U does not divide the debt payment for states between S_a and S_b; for these states the payoff is insensitive to the size of the debt-holders' prior claim. This is due to the fact that they are "frozen out" of the bargaining in the A-C model, being purely passive partners. In r and W, however, P does play a role for the intermediate states, although the power of the debt-holders, and thus their payoff, is least in the truncated game which takes account of the voting power of equity.

The essence of the difference is that in U it is $G(S_a)-G(S_b)$ that is divided equally between debt and equity, while in r and W it is $PQ(S_a)$ that is divided. We conclude this section with three diagrams showing geometrically how the gains from investment are divided.



A-C   r   W

[The shaded areas are divided equally between debt and equity; the areas marked E(B) go entirely to equity (debt).]

IV. <u>The more general model, II</u>: State-sensitive optimal investment levels with bank financing.

In this section, we remark that there is no essential loss of generality in allowing the value-of-investment function to achieve its maximum at levels of investment that vary with the state of the world.

We start with a value-of-investment function $V(I:S) - I$, and the majority coalition will solve the problem

$$\max_{I \in R_+} V(I:S) - I$$

the optimal level of investment being denoted I(S). The "indirect value function" is:

$$V^*(S) = V(I(S):S) - I(S)$$

We shall still retain the assumption that $V^*(S)$ is increasing in S, so that the "breakeven" states for the all equity firm and the "atomistic" levered firm a la A-C have their usual definitions:

$$S_a^* \text{ satisfies } V^*(S_a^*) = 0$$

$$S_b^* \text{ satisfies } V^*(S_b^*) = P$$

To facilitate expressing the values, we modify the definition of G(S):

$$G^*(S) = \int_S^1 q(S)V^*(S)dS$$

and we can now write the three values as:

$$\text{A-C: } U^*(C) = \tfrac{1}{2}[G^*(S_a^*)+G^*(S_b^*)-PQ(S_b^*)]E(C) + \tfrac{1}{2}[G^*(S_a^*)-G^*(S_b^*)+PQ(S_b^*)]B(C)$$

$$\text{untruncated r: } r^*(C) = [G^*(S_a^*)-PQ(S_a^*)]E(C) + [PQ(S_a^*)]B(C)$$

$$\text{truncated r: } W^*(C) = [G^*(S_a^*) - \tfrac{P}{2}Q(S_a^*)]E(C) + \tfrac{P}{2}Q(S_a^*)B(C)$$

V. <u>The more general model, III</u>: Debt-financed, state-dependent optimal investment.

In this section, we examine the effect of requiring that the new investment be financed out of debt capital. In the first instance, we suppose that this capital must be drawn from the existing debt capital, so that a coalition C has at its disposal funds of PB(C) to invest. To simplify matters, we shall normalize investment levels as well, so that

feasible investments for the firm as a whole lie between 0 and 1, thus suppressing P.

Assuming that these monies are denumerated in present dollars, we can write the revised characteristic function of the coalition C in state S as:

$$w(C:S) = \begin{cases} 0 \text{ if } E(C) < \frac{1}{2} \\ [V(B(C),S)-B(C)]E(C)+B(C) \quad \text{if } E(C) \geq \frac{1}{2} \text{ and } B(C) < I(S) \\ V^*(S)E(C) + B(C) \quad \text{otherwise} \end{cases}$$

This function is truncated twice, since both debt- and equity-holders can affect the coalition's ability to undertake optimal-scale investment.

There are two cases, depending on whether $I(S) \gtrless \frac{1}{2}$. In Case I, $I(S) < \frac{1}{2}$, so that any coalition near the diagonal with enough votes to choose an investment plan has enough liquidity to make the optimal scale of investment. In Case II there is a non-null collection of coalitions which can vote for investment but only up to the level of their resources.

The value in Case I is fairly simple, and agrees with the value of the game without this liquidity constraint:

$$\Psi w(C:S) = [V^*(S) + \tfrac{1}{2}]E(C) + \tfrac{1}{2}B(C)$$

This is a sensible formula since with the present normalization the value of the firm's combined debt and equity is $V^*(S) + 1$. Thus, the payment to the debt-holders is divided 50:50.

Case II is more subtle. To analyze this game, we must exploit the linearity of the Shapley value. Let us define two component games, w_1 and w_2 as follows:

$$w_1(C:S) = [V(B(C),S) - B(C)]E(C) + B(C)$$

$$w_2(C:S) = V^*(S)E(C) + B(C)$$

For coalitions along the diagonal, the original game w can be represented as the sum of two games: the game w_1 truncated below $E(C) = \frac{1}{2}$, and the game $w_2 - w_1$ truncated below $B(C) = I(S)$. Since the value of the sum is the sum of the values, we obtain

$$\partial_C w_1(t,t) = [V(t,S) - t]E(C) + [tV'(t,S) + 1 - t]B(C)$$

$$\partial_C(w_2-w_1)t,t) = [V(I(S),S)-I(S)-V(t,S)+t]E(C)+[t-tV'(t,S)]B(C)$$

So the values of the truncated versions of these component games, denoted w_1^t and $(w_2-w_1)^t$, respectively, are:

$$\Psi w_1^t = [\tfrac{1}{2}V(\tfrac{1}{2},S)+\overline{V}(\tfrac{1}{2},S)- \tfrac{1}{8}]E(C)+[V(1,S)- \tfrac{1}{2}V(\tfrac{1}{2},S)+ \tfrac{1}{8} - \overline{V}(\tfrac{1}{2},S)]B(C)$$

$$\Psi(w_2-w_1)^t = [(1-I(S))(V(I(S),S)-I(S) + \tfrac{1}{2} - \tfrac{1}{2}I^2(S)-\overline{V}(I(S),S)]E(C)$$

$$+[\tfrac{1}{2} - \tfrac{1}{2}I^2(S)-V(1,S)+I(S)V(I(S),S)+\overline{V}(I(S),S)]B(C)$$

where

$$\overline{V}(t,S) = \int_t^1 V(t,S)dt$$

Thus, the value of w is

$$\Psi w(C) = [\tfrac{1}{2}V(\tfrac{1}{2},S)+ \tfrac{3}{8} +V(I(S),S)-I(S)-I(S)V(I(S),S)+ \frac{I^2(S)}{2}$$

$$+ \int_{1/2}^{I(S)} V(t,S)dt]E(C)+[\tfrac{5}{8} - \tfrac{1}{2}V(\tfrac{1}{2},S)+I(S)V(I(S),S)$$

$$- \frac{I^2(S)}{2} - \int_{1/2}^{I(S)} V(t,S)dt]B(C)$$

It is easily checked that the value of the entire firm is $V(I(S),S)-I(S)+1$. To obtain the overall value of the firm as of $t = 0$ we merely integrate over states S with the appropriate contingent funds price. This is done in the First Appendix, where the entire analysis is summarized and the shadow prices of debt and equity presented for each approach.

VI. Some Game-Theoretic Issues

In this section, we address four issues. First, we discuss the diagonal formula for the value, giving some sufficient conditions for its validity, and showing that it is still valid when the measures on which the game is based are not independent, so long as they are non-atomic measures. Second, we discuss the general issue of games with discontinuities along the diagonal. One important class of games with this feature is "truncation games" of the sort used here: the range of the component measures is divided into a number of more-or-less regular regions, and a different characteristic function is used in each one. Under certain conditions, the value of the composite game can still be calculated by a modified version of the diagonal formula. This device allowed us to deal with voting stock and capitalization limits, and can also be used to incorporate the effects of other forms of corporate

finance: preferred stock; convertibles; options and the like. The details of these applications will form the subject of a subsequent paper. Third, we shall discuss the relation between these "value-based" results and the core of the underlying game. In their third section, A-C make much of the fact that, with three or more classes of asset-holders, the core may be empty. We shall present conditions that guarantee the non-emptiness of the core, and discuss the meaning of the value when the core is empty. In brief, we shall argue that the reasons leading one to use the value are unaffected by the emptiness of the core.

As mentioned above, one can often express the Shapley value as the integral of a certain derivative. In the case of a "vector measure game", where

$$V(C) = f(\mu_1(C),\ldots,\mu_n(C)) = f(\mu(C))$$

for each coalition C, where the μ_i are nonatomic measures*, we have the formula:

$$V(C) = \int_0^1 f_C(t\cdot\mu(I))dt$$

where f_C denotes the derivative of the function f in the direction $\mu(C)$. In the case where the range of the vector measure μ has dimension n, so that none of the component measures is a function of the other n-1 we can restate this formula as

$$V(C) = \sum_{i=1}^{n} \mu_i(C) \int_0^1 f_i(t\mu(I))dt$$

where f_i is the partial derivative of f w.r.t. its i^{th} argument. In general, if the measures are correlated enough to reduce the dimension of

the range of μ, then there exists another vector measure, ν, whose components are linear combinations of the μ_i, s.t. the above formula is valid when the μ's are replaced with ν's. Of course, we require that the function f be continuously differentiable on the range of μ.

At any rate, this formula tells us that for such a game the values are always linear combinations of the underlying measures. However, the intuition behind the diagonal formula may be a bit unclear, since it seems to suggest that the only thing affecting the value is the behavior of the characteristic function at diagonal coalitions: scaled down versions of the coalition of the whole. It might seem that these diagonal coalitions might not be in the range of μ, or that the Law of Large Numbers might involve the "most probable" coalitions being off-diagonal coalitions. However, since the measures are nonatomic, which means that for every coalition C with $\mu_i(C) > 0$, there exists a subcoalition $C' \subset C$ with $\mu_i(C) > \mu_i(C') > 0$, we can apply Lyapunov's Theorem. This result states that the range of a nonatomic vector-valued measure is convex and compact; since it contains $(0,0,\ldots,0) = \mu(\emptyset)$ and $(\mu_1(I),\ldots,\mu_n(I))$, it must contain all points of the form $(t\mu_1(I),\ldots,t\mu_n(I))$ for $t \in [0,1]$, which is the diagonal. Moreover, the strong LLN tells us that these are indeed the most probable coalitions, even though one cannot formulate the value for nonatomic games precisely in terms of random orders. Finally, it should be noted that while the diagonal may well have nonempty intersection with the boundary of the range of μ, all of the directional derivatives of coalitions C will be evaluated along directions "pointing into" the range of μ.

Examples of sufficient conditions for the value to enjoy the diagonal property include the following:

i) if $V = f \circ \mu$, where μ is a normalized, nonnegative nonatomic measure and f is a function of bounded variation with $f(0) = 0$, continuous at 0 and 1, then V enjoys the diagonal property: its value is completely determined by the behavior of f near the diagonal.

ii) if V belongs to the span of the games defined above, it has the diagonal property.

iii) if V belongs to the closure (in the variation norm) of the space of polynomials of nonatomic measures, it has the diagonal property.

iv) if the value is continuous in the variation norm (as an operator on a reproducing subspace of BV) then it has the diagonal property.

A second issue involves games with discontinuities on or near the diagonal. The diagonal formula gives such a simple method of calculating the value that it would be very convenient if we had a formula that allowed us to derive the values of such patchwork games from their component values.

The simplest class of such games are the truncation games. For simplicity, we will limit attention to vector measure games, so the range of coalitions we have to consider has finite dimension. Let $V = f \circ \mu$ be such a game, and let $V(i,t)$ be the game defined by

$$V(i,t)(C) = \begin{cases} 0 \text{ if } \mu_i(C) < t \\ V(C) \text{ otherwise} \end{cases}$$

Moreover, suppose that $V \in pNAD \cap pNA'$, where pNAD is the space pNA + DIAG, pNA is the closure in the variation norm of the space of polynomials of NA

measures, DIAG is the space of functions of bounded variation vanishing in a neighborhood of the diagonal, and pNA' is the closure of the space of polynomials of NA measures in the sup norm. Then there is a μ-value for V(i,t), and it is given by

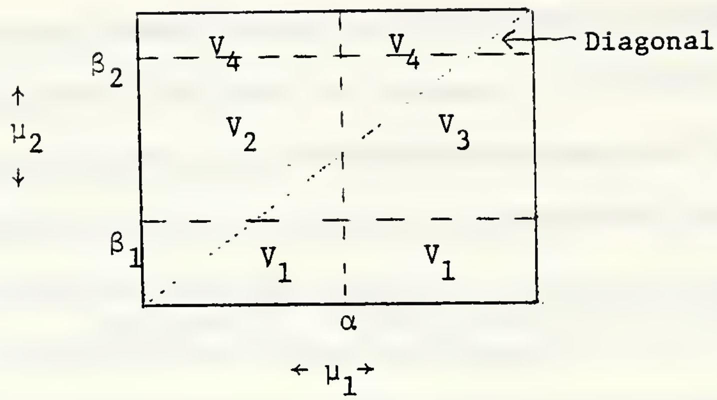
$$\Psi V(i,t)(C) = V^*(t\chi_I)\mu_i(C) + \int_t^1 \partial_C V^*(t\chi_I)dt$$

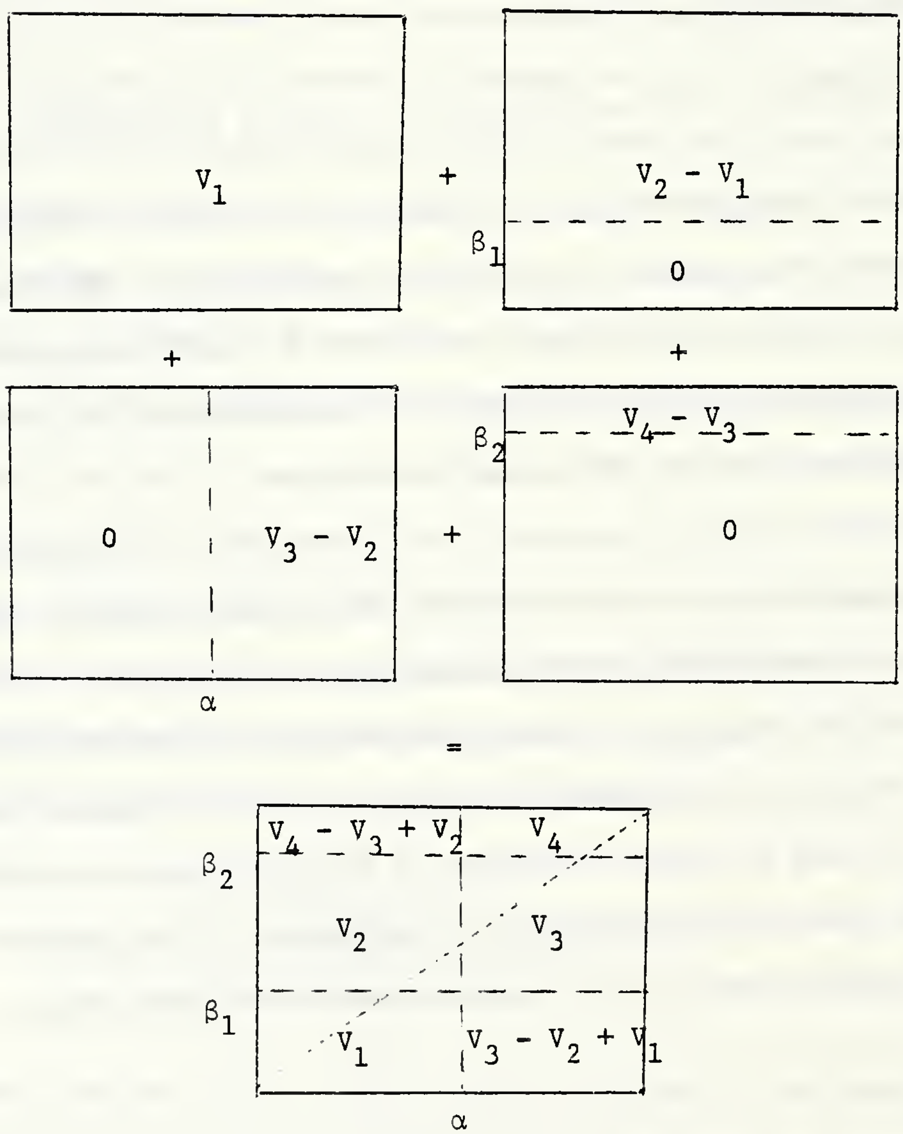
where V* is the extension of V to the space of "ideal set functions" and ∂_C denotes the directional derivative in the direction C. In our vector measure case, this formula becomes

$$\Psi V(i,t)(C) = f^\circ(t\cdot\mu(*))\mu_i(C) + \sum_{j=1}^{n} \mu_j(C) \int_t^1 f_j(t\cdot\mu(I))dt$$

which has the following meaning: the worth of the "pivotal coalition" which is a t-scale sample of I is divided according to μ_i, the truncating measure. For the rest of the game, we proceed as usual, integrating the partial derivatives f_j up the diagonal, and combining them so as to obtain the integral of the directional derivatives.

Now suppose that we have several such truncations. We can combine them using the linearity of the value and the diagonal property. Consider the game described by the following figure. It depends on two measures, and has been built up using four different underlying games. Since our eligible spaces of games are algebras, we can decompose this into a set of sums of simple truncations of eligible games which gives us the right games in a neighborhood of the diagonal. This is done in the second figure below, and the results shown in the third figure.


Diagonal
V4
V4
V2
V3
V1
V1
β2
β1
μ2
α
μ1


V1
+
V2 - V1
β1
0
+
+
0
V3 - V2
α
+
β2
V4 - V3
0
=
V4 - V3 + V2
V4
β2
V2
V3
β1
V1
V3 - V2 + V1
α

From these pictures it is clear that, if V denotes the composite game,

$$\psi V = \psi V_1 + \psi(V_2 - V_1)(2,\beta_1) + \psi(V_3 - V_2)(1,\alpha) + \psi(V_4 - V_3)(2,\beta_2)$$

Letting $V_i = f^i o \mu$ and denoting the first partial derivatives of f^i w.r.t. its first and second arguments by f_1^i and f_2^i, respectively, we may apply the previous formula for the value of truncations of vector measure games to obtain

$$\begin{aligned}\psi V(C) = {} & \mu_1(C)[f^3(\alpha\mu(I)) - f^2(\alpha\mu(I)) + \int_0^{\beta_1} f_1^1(t\mu(I))dt + \int_{\beta_1}^{\alpha} f_1^2(t\alpha(I))dt \\ & + \int_{\alpha}^{\beta_2} f_1^3(t\mu(I))dt + \int_{\beta_2}^{1} f_1^4(t\mu(I))dt] \\ & + \mu_2(C)[f^2(\beta_1\mu(I)) - f^1(\beta_1\mu(I)) + f^4(\beta_2\mu(I)) - f^3(\beta_2\mu(I)) \\ & + \int_0^{\beta_1} f_2^1(t\mu(I))dt + \int_{\beta_1}^{\alpha} f_2^2(t\mu(I))dt + \int_{\alpha}^{\beta_2} f_2^3(t\mu(I))dt \\ & + \int_{\beta_2}^{1} f_2^4(t\mu(I))dt\end{aligned}$$

which is just the result we expect. The truncations divide the diagonal into intervals: $[0,\beta_1)$, $[\beta_1,\alpha)$, $[\alpha,\beta_2)$, and $[\beta_2,1]$. The value is the integral of the directional derivative up the diagonal, using each characteristic function in the interval to which it belongs, together with adjustments for the jumps. In these adjustments, the "jump" (for instance $(f^2 - f^1)(\beta_1\mu(I))$ at β_1) is divided according to the measure defining the jump.

Finally, we must deal with the question of what happens when the jump is not defined by a single measure, but takes place across an arbitrary mainfold. As long as the manifold is transverse to the diagonal,

we can write the boundary locally in the form $\sum_{i=1}^{n} \alpha_i \mu_i = \kappa$, where κ is a constant and the α_i are finite scalars, chosen so that their sum is = 1. We remark that this can be done if and only if the manifold is transversal to the diagonal, so that the tangent to the manifold and the tangent to the diagonal span the tangent space of the range of at that point. In this case, the jump is multiplied for coalition C by $\sum_{i=1}^{n} \alpha_i \mu_i(C)$. Of course, this means that the manifold must be continuously differentiable in a neighborhood of the diagonal, so that the game can be written in terms of n+1 measures. Suppose that $V = f \circ \mu$, where μ is an n-vector of NA measures. Now consider the game V' defined by

$$V'(C) = \begin{cases} V(C) \text{ iff } g(\mu(C)) \geq m \\ 0 \text{ otherwise} \end{cases}$$

where $g\colon\ R^n \to R$ is differentiable in a convex neighborhood of the diagonal, at least for values in a neighborhood of m. We can rewrite this game, after observing that dg, the gradient of g, is a linear function. Therefore, $dg \cdot \mu$ is itself a measure, and $V' = V(n+1,m)$, where $\mu_{n+1} = dg \cdot \mu$.

There are several cases where this cannot be done. For example, if g is not differentiable at the diagonal, the implicit measure cannot be assigned unambiguously and the value is not uniquely defined. An example is $V = \min[\mu_1,\mu_2,c]$, where c is a constant, since this game is the same as

$$V(C) = \begin{cases} \min[\mu_1,\mu_2] \text{ if } \min[\mu_1,\mu_2] \leq c \\ c \text{ otherwise} \end{cases}$$

and it is well-known (cf Aumann-Shapley, p. 140) that this does not have a nice value. Another case is provided by functions g with the property that dg is not transverse to the diagonal at the jump, since the measure will be degenerate at that point.

The next question relates to the emptiness of the core. Since the value reflects the expected utility of taking part in the game, it is defined independently of core considerations, and retains its interpretation as the unique outcome satisfying the conditions of equity and efficiency spelled out in the axioms defining the value. If the core is empty, then any efficient allocation can be prevented by a coalition acting in its own interest. However, the value is more prospective, since it abstracts from the question of which coalition will form. For this reason it is best suited to <u>valuation</u> questions, where the expected worth of the firm as of t=0 is of interest. Nonetheless, it is interesting to know when the core is nonempty. For general nonatomic games, the answer was found by Schmeidler. Let V be a nonatomic game and define

$$|V| = \sup \sum_{i=1}^{n} a_i V(C_i)$$

where the sup runs over all finite sequences of nonnegative weights a_i and coalitions C_i with the following properties:

i) $\overset{n}{\underset{i=1}{}} C_i = I$

ii) for each t I, $\sum_{i=1}^{n} a_i \chi_{C_i}(t) = 1$

where $\chi_C(t)$ is the indicator function of coalition t; equal to 1 if t C and 0 otherwise. Then Schmeidler's result is that the core is nonempty iff $|V| = V(I)$.

REFERENCES

Aivazian, W. and Callen, J., "Corporate Leverage and Growth, Journal of Financial Economics, 1980.

Myers, S., "Determinants of Corporate Borrowing," Journal of Financial Economics, 1977.

M/D/400

APPENDIX

Brief Summary of Characteristic Functions and Values (taking P = 1)

I. Characteristic Functions:

$$V_0(E,B:S) = \begin{cases} 0 \text{ if } S < S_a \\ [V(I(S),S) - I(S)]\cdot\min[E,B] \text{ if } S \quad [S_a,S_b] \\ [V(I(S),S) - I(S) - 1]\cdot E + B \text{ if } S > S_b \end{cases}$$

$$V_1(E,B:S) = \begin{cases} 0 \text{ if } S < S_a \\ [V(I(S),S) - I(S) - 1]\cdot E + B \text{ if } S \geq S_a \end{cases}$$

$$V_2(E,B:S) = \begin{cases} 0 \text{ if } S < S_a \text{ of } E < \frac{1}{2} \\ [V(I(S),S) - I(S) - 1]\cdot E + B \text{ otherwise} \end{cases}$$

$$V_3(E,B:S) = \begin{cases} 0 \text{ if } S < S_a \text{ or } E < \frac{1}{2} \\ [V(B,S) - B]\cdot E + B \text{ if } S \geq S_a,\ E \geq \frac{1}{2},\ \underline{\text{and}}\ B < I(S) \\ [V(I(S) - I(S)]\cdot E + B \text{ otherwise} \end{cases}$$

where

$$V(I(S_a),S_a) = I(S_a)$$

$$V(I(S_b),S_b) = I(S_b) + 1$$

II. Values

These values are linear combinations of the equity measure, E(C) and the debt measure D(C) of the coalition C, so we shall present the "shadow prices" or shares accruing to equity and to debt separately. These are given in the form of $t = 0$ present values, with q(S) being the price at $t = 0$ of $1.00 at $t = 1$ contingent on the occurrence of state S. Other simplifying definitions we shall use are:

$$I(S_c = \frac{1}{2}$$

$$Q(S) = \int_S^1 q(S)dS$$

$$G(S) = \int_S^1 q(S)(V(I(S),S) - I(S))dS$$

A. Shadow prices of equity

V_0: $\frac{1}{2}[G(S_a) + G(S_b) - Q(S_b)]$

V_1: $[G(S_a) - Q(S_a)]$

V_2: $[G(S_a) - \frac{1}{2}Q(S_a)]$

V_3: there are two cases to consider, depending on whether $S_c \gtrless S_a$.

In Case I, $S_c < S_a$ and the shadow pirce of equity is

$$\frac{1}{2}\int_{S_a}^1 q(S)V(\tfrac{1}{2},S)dS + \tfrac{3}{8}Q(S_a) + G(S_a) - \int_{S_a}^1 q(S)I(S)[V(I(S),S) - \frac{I(S)}{2}]dS$$
$$+ \int_{S_a}^1 q(S) \int_{1/2}^{I(S)} V(t,S)dt^{dS}$$

In Case II, $S_c \geq S_a$ and the shadow price of equity is

$$\int_{S_c}^{S_c} q(S)[V(I(S),S) - I(S) - \tfrac{1}{2}]dS + \frac{1}{2}\int_{S_c}^{1} q(S)V(\tfrac{1}{2},S)dS + \frac{3}{8}Q(S_c) + G(S_c)$$
$$- \int_{S_c}^{1} q(S)I(S)[V(I(S),S) - \tfrac{I(S)}{2}]dS + \int_{S_c}^{1} q(S)\int_{1/2}^{I(S)} V(t,S)dtdS$$

B. <u>Shadow Price of Debt</u>

V_0: $\frac{1}{2}[G(S_a) - G(S_b) + Q(S_b)]$

V_1: $Q(S_a)$

V_2: $\frac{1}{2}Q(S_a)$

V_3: in Case I, with $S_c < S_a$, we get

$$\frac{5}{8}Q(S_a) - \frac{1}{2}\int_{S_a}^{1} q(S)V(\tfrac{1}{2},S)dS + \int_{S_a}^{1} q(S)I(S)[V(I(S),S) - \tfrac{I(S)}{2}]dS - \int_{S_a}^{1} q(S)\int_{1/2}^{I(S)} V(t,S$$

and in Case II, with $S_c \geq S_a$, we obtain

$$\int_{S_a}^{S_c} \frac{q(S)}{2}dS + \frac{5}{8}Q(S_c) - \frac{1}{2}\int_{S}^{1} q(S)[V(\tfrac{1}{2},S) + I(S)[I(S) - 2V(I(S),S)]]dS$$
$$- \int_{S_c}^{1} q(S)\int_{1/2}^{I(S)} V(t,S)dtdS$$